

Formalizing Results in Anabelian Geometry (Some Baby Steps)

Adam Topaz

University of Alberta

January 6, 2021

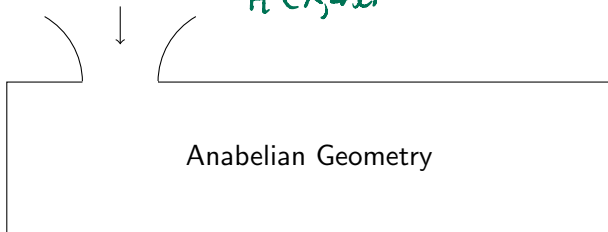
It. w

Colin MacDonnell.

What?

object: K, X

$\text{Gal}(\bar{K}/K), \pi_1^{\text{ét}}(X, \bar{a}),$
Topological Data
 $H^*(X, \mathbb{Z}_\ell).$



Arithmetic/Geometric Data

$K, X.$

Why?

1. Originated from Grothendieck's *Esquisse d'un Programme*.
2. Study solns to polynomial equations using “topological” tools.
3. Interaction between arith and geom highlighted in $\pi_1^{\text{ét}}$.
4. This can lead to new insight.

Mandell Conj.

Why?

It's Fun!

+ interesting.

Why formalize?

1. Work based on a handful of (relatively elementary, but very deep) technical results.
2. No low-hanging fruit. Progress made incrementally.

3. Constructions vs. functoriality.

$$G_K \rightsquigarrow K$$

Hodas.

↙ if $G_K \cong G_L \Leftrightarrow K \cong L$

$$\text{Isom}(K, L) \xrightarrow{\cong} \text{Isom}(G_L, G_K)$$

4. Interaction between different areas of pure math.

Neukirch-Uchida

Why formalize?

It's Fun!

How?

Valuations are the *bridge* that allows us to move from the “topological data” side to the “arithmetic/geometric” side.

Key Reason: Valuations leave a very distinct and recognizable impression on the structures appearing in the topological side.

Valuation Rings

Definition

Let K be a field. A *valuation ring* (of K) is a subring \mathcal{O} such that for all $x \in K$, one has $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

as in mathlib $\mathcal{O}^{-1} = \mathfrak{o}$.

Some objects associated with \mathcal{O} : $\mathfrak{o} \neq \mathfrak{o} \wedge$.

1. The unit group $\mathcal{O}^\times = \{x : K \mid x \in \mathcal{O} \wedge x^{-1} \in \mathcal{O}\}$.
2. The maximal ideal $\mathfrak{m} = \{x : K \mid x \in \mathcal{O} \wedge x \notin \mathcal{O}^\times\}$.
3. The principal units $1 + \mathfrak{m} := \{x : K \mid x - 1 \in \mathfrak{m}\}$.

\mathcal{O}^\times and $1 + \mathfrak{m}$ are both multiplicative subgroups of K^\times .

$\{x : K \mid x \neq 0\}$.

Examples

|| Nat rings of \mathbb{Q}
= $\{ \mathbb{Z}_{(p)} \mid p \text{ prime} \}$.

Some arithmetic examples:

1. $\mathbb{Z}_{(p)} = \{ \frac{a}{b} \mid b \notin p \cdot \mathbb{Z} \} \subset \mathbb{Q}$.
2. $\mathbb{Z}_p \subset \mathbb{Q}_p$.

Some geometric examples:

3. $k[T]_{(T)} = \{ \frac{f}{g} \mid f(0) \neq 0 \} \subset k(T)$.
4. $k[[T]] \subset k((T))$.

Many other examples:

E.g. $\{ \frac{a}{b} \mid a \in \mathbb{N}, b \in \mathbb{N} \} \subset \mathbb{R}^*$

$$k[[T]] = \left\{ \sum_{n=0}^{\infty} a_n T^n \right\}$$

$$k((T)) = k[[T]] \left[\frac{1}{T} \right].$$

"
completion of
 $k(T)$ wrt the T -adic
val.

Field of rational functions

Valuations in mathlib

A byproduct of the perfectoid project?

$$v(x) = |x|_v.$$

monoids
with zero

Definition

A *valuation* on a commutative ring A is a morphism of monoids

$v : A \rightarrow \Gamma$, where Γ is a totally ordered commutative group with 0, satisfying:

1. $v0 = 0$.
2. $v(x + y) \leq \max(v(x), v(y))$ for all $x, y \in A$.

Valuations usually considered up-to equivalence.

$$\mathcal{O} \longmapsto K^x \xrightarrow{|\cdot|_v} K^x / \mathcal{O}^x \quad \text{if } |x|_v \leq |y|_v \text{ iff } y/x \in \mathcal{O}.$$

$$x \longmapsto x \cdot \mathcal{O}^x.$$

Valuation sub-
rings of K .



Valuations
of K . \sim

$$\{x \mid |x|_v \leq 1\}$$

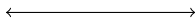
"
 \mathcal{O}_v .



$$|\cdot|_v$$

$$A \rightarrow A/\mathfrak{p} \rightarrow \text{Frac}(A/\mathfrak{p}) \xrightarrow{\cong} \mathbb{Q}$$

Valuations
of A .



$\mathfrak{p} \in \text{Spec } A$ +
valuation on
 $\text{Frac}(A/\mathfrak{p})$.

Example: Number Fields

Thm (Neukirch-Uchida):
if K, L are # fields then
 $K \cong L$ iff $\text{Gal}_K \cong \text{Gal}_L$.

K and L number fields. Given $\phi : \text{Gal}_K \cong \text{Gal}_L$.

1. ϕ induces bijection on *decomposition groups*.

↳ Global class field theory

2. Valuations "parameterized" by decomposition groups.

↳ valuation thy (Approx thms for indep vals).

3. Numerical data determined by decomposition groups.

↳ N p-adic val $D_v \rightarrow P, g, e, f, I_v \subset D_v$ etc...

4. Local correspondence + numerical data $\dots \bar{K}|K \cong \bar{L}|L$.

Local CFT.

Chebotarev Density

Idea: Geometric Function Fields

Bogomolov - Tinkler,
Rep.

$$L = k(X), k = \bar{k}, \dim X \geq 2.$$

Given $H^*(L, \mathbb{Z}_\ell)$, a graded-commutative ring.

1. $H^*(L, \mathbb{Z}_\ell) \dots \hat{K}_*(L).$

↳ Bledar-Kodric Conj. (Merkurjev Suslin)

2. Obtain information about "geometric" valuations.

↳ This partially formalised

3. $\text{Vals} + \hat{K}_1(L) = \hat{L}^\times + \hat{K}_*(K) \dots \boxed{L^\times/k^\times + \text{comb. geom.}}$

4. $L^\times/k^\times + \text{comb. geom.} \dots (L, +, \times).$

↳ fundamental thm of proj. geometry

F field.

$$K_*(F) := \frac{T_*(F^X)}{\langle x \otimes (1-x) \rangle} \dashrightarrow \text{Val. rings of } F$$

Partially Formalized

$$\frac{T_*(F^X)}{\langle x \otimes (1-x) \rangle} = \frac{\text{the tensor algebra } \checkmark \text{ of the action of } F^X}{\langle n \otimes (1-n) \rangle.}$$

straightening relation \checkmark / $n \in F \setminus \{0,1\}$

(DEMO)

Future

1. Generalize: $\mathbb{Z}/2 \rightsquigarrow \mathbb{Z}/p, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}_\ell$.
2. *The Fundamental Theorem of Projective Geometry.*

(eventually. . .)

3. Connect with the Galois-theoretic side of the story:
 $\pi_1^{\text{ét}}, H^*(X, \mathbb{Z}_\ell), \text{Gal}(\bar{K}|K), \text{Kummer theory, etc.}$