Formalizing Results in Anabelian Geometry (Some Baby Steps)

Adam Topaz

University of Alberta

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Jt. w/ Colter MacDonald.

What?

object: K, X

$\mathrm{Gal}(\overline{K}|K)$, $\pi_1^{et}(X,x)$.

Topological Data

$H^*(X,\mathbb{Z}_\ell)$.

↓

Anabelian Geometry

↓

Arithmetic/Geometric Data

K, X.

Why?

1. Originated from Grothendieck's *Esquisse d'un Programme*.
2. Study solns to polynomial equations using "topological" tools.
3. Interaction between arith and geom highlighted in $\pi_1^{\text{ét}}$.
4. This can lead to new insight.

Mordell Conj.

Why?

It's Fun!

+ Interesting.

Why formalize?

1. Work based on a handful of (relatively elementary, but very deep) technical results.
2. No low-hanging fruit. Progress made incrementally.
3. Constructions vs. functoriality.

   if $G_K \cong G_L \iff K \cong L$

   $\mathrm{Isom}(\cancel{K},\bar{L}) \cong \mathrm{Isom}(G_L, G_K)$

   $G_K \rightsquigarrow K$

4. Interaction between different areas of pure math.

Hoshi.

Neukirch–Uchida

Why formalize?

It's Fun!

How?

Valuations are the *bridge* that allows us to move from the "topological data" side to the "arithmetic/geometric" side.

Key Reason: Valuations leave a very distinct and recognizable impression on the structures appearing in the topological side.

Valuation Rings

as in mathlib $0^{-1} = 0$.

Definition

Let K be a field. A *valuation ring* (of K) is a subring $\mathcal{O}$ such that for all $x \in K$, one has $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Some objects associated with $\mathcal{O}$: $x \neq 0 \wedge$.

1. The unit group $\mathcal{O}^\times = \{x : K \mid x \in \mathcal{O} \wedge x^{-1} \in \mathcal{O}\}$.
2. The maximal ideal $\mathfrak{m} = \{x : K \mid x \in \mathcal{O} \wedge x \notin \mathcal{O}^\times\}$.
3. The principal units $1 + \mathfrak{m} := \{x : K \mid x - 1 \in \mathfrak{m}\}$.

$\mathcal{O}^\times$ and $1 + \mathfrak{m}$ are both multiplicative subgroups of $K^\times$.

$K^\times = \{x : K \mid x \neq 0\}$.

Examples

Val rings of $\mathbb{Q}$
$= \{ \mathbb{Z}_{(p)} \mid p \text{ prime} \}$.

Some arithmetic examples:

1. $\mathbb{Z}_{(p)} = \{\frac{a}{b} \,|b \notin p \cdot \mathbb{Z}\} \subset \mathbb{Q}$.
2. $\mathbb{Z}_p \subset \mathbb{Q}_p$.

$k[[T]] = \{ \sum_{n=0}^{\infty} a_n T^n \}$

$k((T)) = k[[T]][\frac{1}{T}]$.

$T \cdot k[T]$.

Some geometric examples:

3. $k[T]_{(T)} = \{\frac{f}{g} \mid f(0) \neq 0\} \subset k(T)$.
4. $k[[T]] \subset k((T))$.

" completion of $k(T)$ at the T-adic val.

field of rational functions

Many other examples:

e.g. $\{x \mid \exists n \in \mathbb{N}, |x| \leq n\} \subset \mathbb{R}^*$

Valuations in `mathlib`

A byproduct of the perfectoid project?

$v(x) = |x|_v.$

monoids with zero

Definition

A *valuation* on a commutative ring A is a morphism of monoids
$|\cdot|$ $v : A \to \Gamma$, where Γ is a totally ordered commutative group with 0, satisfying:

1. $v0 = 0$.
2. $v(x + y) \leq \max(v(x), v(y))$ for all $x, y \in A$.

Valuations usually considered up-to equivalence.

$|x|_v \leq |y|_v$ iff

$$\mathcal{O} \longmapsto K^\times \xrightarrow{|\cdot|_v} K^\times/\mathcal{O}^\times \qquad y/x \in \mathcal{O}.$$

$$x \longmapsto x \cdot \mathcal{O}^\times.$$

Valuation subrings of K. $\longleftrightarrow$ Valuations of K. $/\sim$.

$$\{x \mid |x|_v \leq 1\} \longleftarrow |\cdot|_v$$

$$= \mathcal{O}_v.$$

$$A \to A/\mathfrak{p} \to \mathrm{Frac}(A/\mathfrak{p}) \xrightarrow{v} \Gamma \quad \longmapsto \quad v$$

Valuations of A. $\longleftrightarrow$ $\mathfrak{p} \in \operatorname{Spec} A$ + valuation on $\mathrm{Frac}(A/\mathfrak{p})$.

Example: Number Fields

Thm (Neukirch-Uchida): if K, L are # fields then $K \cong L$ iff $\mathrm{Gal}_K \cong \mathrm{Gal}_L$.

K and L number fields. Given $\phi : \mathrm{Gal}_K \cong \mathrm{Gal}_L$.

1. ϕ induces bijection on *decomposition groups*.
   ↳ Global class field theory
2. Valuations "parameterized" by decomposition groups.
   ↳ valuation thy (Approx thms for indep vals).
3. Numerical data determined by decomposition groups.
   ↳ $\forall$ padic val $D_v \rightsquigarrow p, q, e, f, I_v \subset D_v$ etc...
4. Local correspondence + numerical data $\cdots$ $\bar{K}|K \cong \bar{L}|L$.

Local CFT.

Chebotarev Density

Idea: Geometric Function Fields

Bogomolov-Tschinkel Rep.

$L = k(X)$, $k = \bar{k}$, $\dim X \geq 2$.
Given $\mathrm{H}^*_{\text{ét}}(L, \mathbb{Z}_\ell)$, a graded-commutative ring.

Thm. Rost-Voevodsky

1. $\mathrm{H}^*(L, \mathbb{Z}_\ell) \quad \cdots \quad \widehat{\mathrm{K}}_*(L)$.

   ↳ Bloch-Kato Conj. (Merkurjev Suslin)

2. Obtain information about "geometric" valuations.

   ↳ This partially formalized

3. Vals $+ \widehat{\mathrm{K}}_1(L) = \widehat{L^\times} + \widehat{\mathrm{K}}_*(K) \cdots \boxed{L^\times/k^\times + \text{comb. geom.}}$

4. $L^\times/k^\times$ + comb. geom. $\cdots$ $(L, +, \times)$.

   ↳ Fundamental thm of proj geometry

F field.

$$K_*(F) := \boxed{\frac{T_*(F^\times)}{\langle x\otimes(1-x)\rangle}} \dashrightarrow \text{Val. rings of } F$$

Partially Formalized

$$\frac{T_*(F^\times)}{\langle x\otimes(1-x)\rangle} = \frac{\text{the tensor algebra (over } \mathbb{Z}\text{.) of the abelian gp } F^\times}{\langle x \otimes (1-x)\rangle}.$$

Steinberg relation $/ x \in F \setminus \{0,1\}$

(DEMO)

Future

1. Generalize: $\mathbb{Z}/2 \rightsquigarrow \mathbb{Z}/p$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_\ell$.

2. *The Fundamental Theorem of Projective Geometry.*

   (eventually...)

3. Connect with the Galois-theoretic side of the story: $\pi_1^{\text{ét}}$, $\mathrm{H}^*(X, \mathbb{Z}_\ell)$, $\mathrm{Gal}(\bar{K}|K)$, Kummer theory, etc.