Measure Theory

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Measure Theory in Lean

Measure theory in Lean currently contains 10k lines of code + 2k lines of comments (slightly less than 4% of mathlib).

Some highlights:

- Yury Kudryashov: FTC-1, Jensen's inequality, ...
- Benjamin Davidson: FTC-2 (as of 12 hours ago)
- Rémy Degenne: L^p-spaces and Hölder's inequality.
- Markus Himmel: Borel-Cantelli lemma (one direction)
- Martin Zinkevich: working towards the Radon–Nikodym theorem
- Sebastien Gouezel: refactor of the Bochner integral
- Me: Haar Measure, Fubini's Theorem

Much of the original library was built by Johannes Hölzl, Mario Carneiro and Zhouhang Zhou.

Refresher on Measure Theory

Definition

A σ-algebra Σ on X is a collection of subsets of X that contains the empty set, is closed under complement and countable unions.

Definition

If Σ is a σ-algebra on X, then a measure on Σ is a function $\mu : \Sigma \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and μ is *countably additive*: For pairwise disjoint sets $\{A_i\}_i$ we have

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

Definition

An outer measure on X is a monotone function $m : \mathcal{P}(X) \to [0, \infty]$ such that $m(\varnothing) = 0$ and m is *countably subadditive*: for any sets $\{A_i\}_i$ we have

$$m\left(\bigcup_{i\in\mathbb{N}}^{\infty} A_i\right) \le \sum_{i\in\mathbb{N}} m(A_i).$$

Design Decisions

- A σ-algebra on a type X is given by the typeclass `[measurable_space X]`.
- We define a measure as an outer measure with two extra properties:
  - It is countably additive on measurable sets
  - For a (non-measurable) set s

$$\mu(s) = \prod_{\substack{s \subseteq t \\ t \text{ measurable}}} \mu(t)$$

  This means that we can nicely evaluate a measure on any set, not just measurable ones.
- You can talk about measures by either having `example {α} [measurable_space α] (μ : measure α)` or `example {α} [measure_space α]`.
  In the second case, the measure is called `volume`. All results about arbitrary measures are written using the first option.

Lebesgue integral

The library contains two integrals.

Definition

If $g : X \to [0, \infty]$ is a simple function (a function with finite range), we can define

$$\int g \, \mathrm{d}\mu = \int g(x) \, \mu(\mathrm{d}x) = \sum_{y \in g(X)} \mu(g^{-1}\{y\}) \cdot y.$$

If $f : X \to [0, \infty]$ is any function, we can define the (lower) Lebesgue integral of f as the supremum of $\int g \, \mu(\mathrm{d}x)$ for all simple $g \leq f$ (pointwise).

Bochner integral

Definition

If E is a (second-countable) Banach space then we call a function $f : X \to E$ (μ-)integrable if

$$\int \|f\| \, \mathrm{d}\mu < \infty$$

Definition

We define $L^1(X, E; \mu)$ to be the μ-integrable functions $f : X \to E$ modulo a.e. equality. So $f \sim g$ iff $\mu\{x \mid f(x) \neq g(x)\} = 0$.

Definition

The Bochner integral $\int f \, \mathrm{d}\mu \in E$ is defined first for simple L^1 functions. The simple functions are dense $L^1(X, E; \mu)$, so we can continuously extend it to all integrable functions.

Product measures

Definition

A measure μ on X is σ-finite if X can be covered by a countable collection of measurable sets $\{A_i\}_i$ such that $\mu(A_i) < \infty$.

If μ is a σ-finite measure on X and ν a σ-finite measure on Y then we can define the product measure $\mu \times \nu$ on $X \times Y$. It can be defined as

$$(\mu \times \nu)(A) = \int_X \nu\{y \mid (x,y) \in A\} \, \mu(\mathrm{d}x).$$

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Fun fact: using the Giry monad, it can be defined as

$$\begin{aligned} \mu \times \nu = \mathrm{do}\ & x \leftarrow \mu, \\ & y \leftarrow \nu, \\ & \mathrm{return}\ (x, y). \end{aligned}$$

Tonelli's theorem

Theorem (Tonelli's theorem)

Let $f : X \times Y \to [0, \infty]$ *be a measurable function (i.e. the preimage of measurable sets under* f *are measurable).*
Then

$$\int_{X\times Y} f \,\mathrm{d}(\mu\times\nu) = \int_X \int_Y f(x,y)\,\nu(\mathrm{d}y)\mu(\mathrm{d}x) = \int_Y \int_X f(x,y)\,\mu(\mathrm{d}x)\nu(\mathrm{d}y),$$

and all the functions in the integrals above are measurable.

Fubini's theorem

Theorem (Fubini's theorem for the Bochner integral)

Let E be a second-countable Banach space and $f : X \times Y \to E$ be an integrable function (i.e. $\int_{X \times Y} \|f\| \mathrm{d}(\mu \times \nu) < \infty$).
Then

$$\int_{X \times Y} f \,\mathrm{d}(\mu \times \nu) = \int_X \int_Y f(x,y)\, \nu(\mathrm{d}y) \mu(\mathrm{d}x) = \int_Y \int_X f(x,y)\, \mu(\mathrm{d}x) \nu(\mathrm{d}y),$$

Moreover, all the functions in the integrals above are measurable.

Remark. $f : X \times Y \to E$ is integrable iff the following two conditions hold:

- for almost all $x \in X$ the function $y \mapsto f(x,y)$ is integrable;
- The function $x \mapsto \int_Y \|f(x,y)\|\, \nu(\mathrm{d}y)$ is integrable.

Remarks

- I followed Isabelle's formalization for some key lemmas in the proof of Fubini's theorem.
- Since yesterday we can take finite products of measures.
- The following two induction principles are useful:
    1. To show a property P for all measurable functions $X \to [0, \infty]$ it is sufficient to show it for $c \cdot \chi_A$, and that the property is closed under addition and countable monotone supremum.
    2. To show a property P for all integrable functions $X \to E$ it is sufficient to show it for $c \cdot \chi_A$, that the property is closed under addition and a.e.-equality and that $\{f \in L^1(X, E; \mu) \mid P(f)\}$ is a closed set.

Regular measures

Definition

A measure μ is regular if the following properties hold.

- if K is compact then $\mu(K) < \infty$;
- μ is outer regular: if A is measurable, then
$$\mu(A) = \inf_{U \supseteq A \text{ open}} \mu(U);$$
- μ is inner regular: if U is open, then
$$\mu(U) = \sup_{K \subseteq U \text{ compact}} \mu(K).$$

Invariant measures

Definition

A topological group is a group $(G, \cdot, ^{-1})$ that is also a topological space such that the multiplication $\cdot : G \times G \to G$ and inversion $^{-1} : G \to G$ are continuous.

Definition

A (left) invariant measure on a topological group G (equipped with the Borel σ-algebra) is a measure μ such that for all $g \in G$ and all measurable A we have

$$\mu(gA) = \mu(A).$$

Here $gA = \{gh \mid h \in A\}$.

Haar measure

Theorem

Given a locally compact[a] Hausdorff[b] topological group G. Then there is a nonzero left invariant regular measure μ on G, called the (left) Haar measure on G. Moreover, if ν is another left Haar measure on G, then ν is a multiple of μ.

[a] every point has a compact neighborhood
[b] every pair of distinct points has a pair of disjoint neighborhoods

Example: $(\mathbb{R}, +)$ is a locally compact Hausdorff group, and the Haar measure of $\mathbb{R}$ is (a multiple of) the Lebesgue measure λ, which is the unique measure with the property that $\lambda([a, b]) = b - a$ for all a and b.

Proof sketch (existence)

- If K is compact and U is open, we can cover K with finitely many left-translates gU for $g \in G$. Define $(K : U)$ to be the least number of left-translates needed.
- Fix any compact set K_0 with non-empty interior. We can approximate the Haar measure of K by $h_U(K) = (K : U)/(K_0 : U)$.
- Let $h(K)$ be the "limit" of this quotient as U becomes a smaller and smaller open neighborhoods of 1.[1]
- We can now define the Haar measure on open sets U as

$$\mu(U) = \sup_{K \subseteq U \text{ compact}} h(K).$$

We can extend it to all measurable sets A by

$$\mu(A) = \inf_{U \supseteq A \text{ open}} \mu(U).$$

[1] The technical details involve infinite products and Tychonoff's theorem.

Uniqueness

- If μ is left invariant, then for each y we have

$$\int_G f(yx)\mu(\mathrm{d}x) = \int_G f(x)\mu(\mathrm{d}x).$$

- Uniqueness can be proven by computing a certain double integral in a smart way, and using Tonelli's theorem to swap the order of integration.
- It is almost `sorry`-free™.

Future work

Some important future directions:

- Using Haar measure we can start on abstract harmonic analysis, Pontryagin duality, and representation theory of locally compact groups.
- Multivariate calculus: Green's theorem and Stokes' theorem.
- Complex calculus: Cauchy's integral formula.

Thank You