Using Lean for teaching

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January 7, 2021

Context

- Title: Computer assisted logic and proofs
- 50 students enrolled in first year of double-major Math and Computer Science, better than average French student but not elite
- Second semester, after one semester of calculus (with ε and δ by teacher but not in exam)
- 12 times 2 hours of computer lab, 2 groups of 25 (other teacher *not* using Lean outside of this course)

Goal

Goal: Improve understanding and production of *traditional* proofs on paper.

Non goals: Lean expertise, type theory, first order logic...

Technology

- Lean + mathlib
- Using VScode or CoCalc or javascript version
- One custom tactic gathering `refl`, `norm_num`, `ring`, `abel`
- Liberal use of `linarith`

Lecture notes

Exemple

> Si une suite u tend vers x_0 et si une fonction f est continue en x_0 alors la suite $f \circ u$ (de terme général $f(u_n)$) tend vers $f(x_0)$.
>
> ```
> example (f u x₀) (hu : limite_suite u x₀)
>   (hf : continue_en f x₀) : limite_suite (f ∘ u) (f x₀) :=
> ```

Démonstration

Notre but est de montrer que, pour tout ε strictement positif, il existe un entier N tel que, pour tout entier n supérieur à N, $|f(u_n) - f(x_0)|$ est inférieur à ε.

```
unfold limite_suite,
```

Soit ε un réel strictement positif.

```
intros ε hε,
```

Par hypothèse de limite sur f, appliquée à cet ε strictement positif, il existe un réel δ strictement positif tel que :

Pour tout x réel, si $|x - x_0| \leq \delta$ alors $|f(x) - y_0| \leq \varepsilon$ (1).

On fixe un tel δ.

```
obtain ⟨δ, δ_pos, Hf⟩ : ∃ δ > 0, ∀ x, |x - x₀| ≤ δ →
|f x - f x₀| ≤ ε,
  exact hf ε hε,
```

Par l'hypothèse de limite sur u, appliquée au réel δ strictement positif que nous venons de fixer, il existe un entier N tel que :

Pour tout entier n, si $n \geq N$ alors $|u_n - x_0| \leq \delta$ (2).

On fixe un tel N.

```
obtain ⟨N, Hu⟩ : ∃ (N : ℕ), ∀ (n : ℕ), n ≥ N → |u n
- x₀| ≤ δ,
  exact hu δ δ_pos,
```

Montrons que N convient à notre objectif.

```
2 goals
f : ℝ → ℝ,
u : ℕ → ℝ,
x₀ : ℝ,
hu : limite_suite u x₀,
hf : continue_en f x₀,
ε : ℝ,
hε : ε > 0
⊢ ∃ (δ : ℝ) (H : δ > 0), ∀ (x : ℝ), |x - x₀| ≤ δ → |f x - f x₀|
≤ ε
```

```
f : ℝ → ℝ,
u : ℕ → ℝ,
x₀ : ℝ,
hu : limite_suite u x₀,
hf : continue_en f x₀,
ε : ℝ,
hε : ε > 0,
δ : ℝ,
δ_pos : δ > 0,
Hf : ∀ (x : ℝ), |x - x₀| ≤ δ → |f x - f x₀| ≤ ε
⊢ ∃ (N : ℕ), ∀ (n : ℕ), n ≥ N → |(f ∘ u) n - f x₀| ≤ ε
```

Mathematical content

- More formal introduction to quantifiers and logic operators but almost only with familiar concrete examples
- Divisibility in $\mathbb{Z}$, functions (injective, surjective, monotone)
- Limits of sequences of real numbers and functions
- $\sup$, $\inf$, subsequences, Bolzano-Weirstrass, Heine

All content became the tutorials project.

Main trick

Carefully select exercises to hide all issues, especially coercions and dependent types (beyond `Prop`)

Effect

First iteration was too hard (students were less prepared than I thought).

Second iteration was much better but Covid interfered and going from Lean to paper is difficult.

Some mistakes in first iteration

- Teaching goal focussing brackets {/} during first iteration.
- Any discussion of differences between set theory and type theory beyond the notation for implication and $x : \mathbb{R}$ instead of $x \in \mathbb{R}$. For instance $A : \mathrm{set}\,\mathbb{R}$ is no problem.

Future

Next iteration starts at end of month. Will probably try verbose custom tactic.

```
-- If f is continuous at x₀ and the sequence u tends to x₀ then the sequence f ∘ u, sending n to
-- f (u n) tends to f x₀
example (f u x₀) (hu : limite_suite u x₀) (hf : continue_en f x₀) :
  limite_suite (f ∘ u) (f x₀) :=
begin
  Let's prove that ∀ ε > 0, ∃ (N : ℕ), ∀ n ≥ N, |(f ∘ u) n - f x₀| ≤ ε,

  Let (ε > 0),

  By hf applied to [ε, h] we obtain
    (δ : ℝ) (δ_pos : δ > 0) (Hf : ∀ (x : ℝ), |x - x₀| ≤ δ → |f x - f x₀| ≤ ε),

  By hu applied to [δ, δ_pos] we obtain
    (N : ℕ) (Hu : ∃ (N : ℕ), ∀ (n : ℕ), n ≥ N → |u n - x₀| ≤ δ),

  Let's prove that N works,

  Let (n ≥ N),

  We apply Hf,

  Let's prove that |u n - x₀| ≤ δ,

  This is Hu n h_1,
end
```